Ising-Bloch transition for the parametric Ginzburg-Landau equation with rapidly varying perturbations

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We study the effects of rapid periodic and stochastic modulations of parameters in systems described by the complex parametric Ginzburg-Landau equation. Amplitude equations, which govern the dynamics of the field averaged over the rapid modulations, are derived. For temporal modulations of the linear detuning the threshold for the transition from Ising to Bloch walls is shifted depending on the strength of the perturbation. In contrast to this, rapid perturbations of the linear gain lead only to a decrease of the amplitude of both wall types leaving the bifurcation point of the Ising-Bloch transition unchanged. Stochastic perturbations of the detuning lead to a Brownian motion of the Bloch wall beyond bifurcation where the velocity is given analytically. All theoretical predictions are confirmed by numerical simulations of the full stochastic Ginzburg-Landau equation.

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I. INTRODUCTION

The influence of noise on the evolution of spatially extended systems described by the Ginzburg-Landau equation attracts a great deal of attention because of their fundamental significance and relevance to many real physical systems. Many investigations have been devoted to symmetry breaking transitions caused by additive and multiplicative noise. The periodic or random modulation of the gain parameter, which may originate from, e.g., the variation of a voltage applied to a liquid crystal, is one typical example. The dependence of the threshold of the symmetry breaking transition on the noise strength, the change of the character of bifurcation, etc., have been investigated by many authors [1–5]. Methods of analysis of stochastic partial differential equations were used for such theoretical investigations; in particular the analysis of stochastic field moments has been applied. In a recent work by Staliunas [6] the influence of spatial randomness on dissipative (cavity) solitons has been investigated. It was shown that the randomness induced by the roughness of the mirrors can stabilize dissipative spatial solitons in nonlinear optical resonators due to the additional damping originated by noise.

In this paper we study the influence of rapidly varying periodic and stochastic perturbations in transversally extended systems near a so-called chirality breaking bifurcation. This type of bifurcation occurs for example in systems that can be modeled by the parametric complex Ginzburg-Landau (GL) equation:

$$F_t = \mu F^* + (\gamma + i\delta)F - |F|^2 F + F_{xx}, \qquad (1)$$

where μ is the parametric forcing parameter with frequency twice that of a Hopf oscillation, γ is the linear gain, and δ denotes the linear detuning [7]. The parametric complex GL equation describes, for example, a degenerate optical parametric oscillator with a frequency limiter acting near antiresonance [8] and the parametric instability in a vertically vibrated layer of granular materials [9]. The parametric forcing breaks the phase invariance ($F \to F \exp i\varphi$) of the usual Ginzburg-Landau equations [10,11] and this gauge symmetry is replaced by the discrete symmetry $F \to -F$. Therefore any nontrivial solution of Eq. (1) has usually a counterpart with a π phase shift compared to the original one. For vanishing detuning ($\delta = 0$) an energy functional E of Eq. (1) exists [7], which allows one to reformulate the parametric GL equation in a gradient form $F_t = -\delta E / \delta F^*$; however, for $\delta \neq 0$ Eq. (1) describes a nongradient system. Domain walls are one prominent example of nonlinear transverse structures of Eq. (1). They can be considered as the transverse connection of two nontrivial plane waves, which are related by the abovementioned, discrete phase symmetry $F \to -F$. The parametric GL equation supports two different kinds of domain walls, namely, the so-called Ising and Bloch walls. In the gradient limit ($\delta = 0$) both types of domain walls can be obtained analytically [7]. The Ising wall has the form

$$\mathrm{Re}\, F_I = \pm \sqrt{\mu + \gamma} \tanh\{[(\mu + \gamma)/2]^{1/2} x\}, \quad \mathrm{Im}\, F_I = 0, \qquad (2)$$

whereas the Bloch wall solution is given by

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$$\mathrm{Re}F_B = \pm\sqrt{\mu+\gamma}\tanh[\sqrt{2\mu}x], \quad \mathrm{Im}F_B = \frac{\sqrt{\gamma-3\mu}}{\cosh(\sqrt{2\mu}x)}. \tag{3}$$

Bloch walls emanate from the Ising wall at the bifurcation point $\gamma_c=\mu/3$. For $\gamma>\mu/3$ the Ising wall destabilizes and stable Bloch walls emerge, i.e., an Ising-Bloch transition occurs. For a gradient system both the Ising and the Bloch walls are at rest, because they obey the parity symmetry $F_{I/B}(x)=-F_{I/B}(-x)$. In the nongradient case ($\delta\neq 0$), Bloch walls exhibit a spontaneous motion, because now the parity symmetry is broken [7]. The motion is induced by an eigenvector, which exactly passes the translational mode of the system at the bifurcation point [11,12].

In this paper we will consider the above parametric GL equation (1) with coefficients variable in time. A laser cavity with the injection of two coherent fields of equal amplitudes and different frequencies is a typical example for such a system. Close to the lasing threshold it can be described by the complex Ginzburg-Landau equation

$$A_t = [\sigma + i\delta(\omega t)]A + (1+i\alpha)A_{xx} - (1+i\beta)|A|^2A + B(\Omega t), \tag{4}$$

where $B(t)$ denotes the external forcing. Among such laser systems a fiber ring laser [14] with filtering exhibits a rather interesting example, where the parameters can be tuned in a wide range in the above equation. In this case A is the field amplitude, t is propotional to the number of circulations of the light in the cavity, and x denotes the normalized time. σ,α, and β represent the parameters describing the small signal gain, the group velocity dispersion, and the refractive nonlinearity, respectively. The cavity detuning $\delta(\omega t)$ depends on the variable t, which, for example, can be realized electro-optically. The parameters originating from filtering (real prefactor of A_{xx}) and from gain saturation (real prefactor of $|A|^2A$) are normalized to unity. By changing the main optical frequencies the strength of dispersion of the system can be varied in a wide range. In particular, for optical frequencies close to the zero dispersion point of the fiber the value of α vanishes and only an effective diffusion due to filtering remains [15]. Similarly, the relation between refractive and absorptive nonlinearities can be adjusted by choice of suitable gain media. For example, in the case of a gain medium with a small saturation power and a small Henry factor, such as a semiconductor quantum dot amplifier, gain saturation is the dominant nonlinear effect. Thus, in what follows we assume that the coefficients α and β are small and can be neglected.

Furthermore, let us consider an external forcing of $B(t)=B\cos(\Omega t)$ where its amplitude is slowly varying in time. Assuming $1\ll\omega\ll\Omega$ (see, e.g., [13]) the field amplitude can be represented in the form of converging series $A=F+\epsilon A_1+\cdots$ where $\epsilon=1/\Omega$ holds. F is a slowly varying function on the scale of $1/\epsilon$ and A_i are rapidly varying functions. After averaging over the frequency Ω we arrive at the parametric GL equation (1) with $\mu=B^2/(2\Omega^2)$ and $\gamma=\sigma-2\mu$.

In order to analytically study the influence of temporal parameter modulations one needs to average the parametric GL equation (1) over these oscillations or fluctuations. A similar problem has been previously investigated for the nonlinear Schrödinger (NLS) equation with rapid modulations of its parameters. An optical fiber transmission line with periodic lumped amplifiers is a spectacular example of such a system. The corresponding NLS equation exhibits a rapidly oscillating, strong amplification term. The averaging of the field over fast oscillations leads to the renormalized NLS equation. The corresponding description is referred to as the guiding-center soliton concept [16]. Analogous considerations for the two-dimensional (2D) NLS equation with a rapidly spatially varying potential result in a renormalized 2D NLS equation too. It represents one option for arresting beam collapse [17]. This stabilization reminds one of the Kapitza stabilization of the inverted pendulum with rapid oscillations of its pivot point [18].

It is worth noting that the common theoretical approach to obtain the threshold value of a symmetry breaking transition in systems with multiplicative noise (e.g., the electrically driven Fréederickz transition in liquid crystals [4]), which is based on the calculation of the first few moments of the linearized equation, leads to the incorrect result for our problem. The reason is the overestimation of events with large deviations, which are actually suppressed by the nonlinear term in the full equation (1). In order to get the correct critical point of the bifurcation from the first moment of the linearized stochastic equation a method has to be applied that excludes contributions from the rare large perturbations [2]. On the other hand one can expect that a consistent accounting for the nonlinearity in the equation for the first moment will as well lead to the correct bifurcation threshold. This equation will be the basis for the analysis of the Ising-Bloch bifurcation under random modulations. For rapid periodic variations of parameters we will derive the averaged GL equation by applying a multiscale technique, similar to the one used in [17,19].

The paper is structured as follows. In Sec. II we present the derivation of the averaged GL equation for rapidly oscillating detuning and linear gain. This equation is applied to analyze the Ising-Bloch transition. Section III is devoted to stochastic parameter fluctuations with respect to Ising-Bloch transitions. Furthermore, the diffusion coefficient for the Brownian motion of Bloch walls is derived. All analytical results are double checked by the numerical simulations of the full parametric GL equation with varying parameters. For the numerical simulations a conventional beam propagation technique is applied using either a fast Fourier transform or a Crank-Nicholson method. The stochastic fluctuations of parameters are modeled by means of a standard random number generator providing a uniform probability distribution, which is transformed into a Gaussian distribution using the Box-Muller method.

II. RAPID PERIODIC MODULATIONS

In order to derive an averaged amplitude equation for rapidly oscillating parameters of Eq. (1) different schemes can be applied, e.g., a multiscale method [19] or a Fourier transform technique [17]. We use the method developed in [17] for the NLS equation with rapidly varying coefficients.

Let us first consider the case of a rapid and harmonic oscillating linear detuning $\delta=\delta(t/\epsilon)=\delta_0+\delta_1\sin(\Omega t)$, $\epsilon^{-1}\sim\Omega\gg 1$. We are looking for solutions of Eq. (1) of the following form:

$$F(x,t)=\overline{F}+A(x,t)\sin(\Omega t)+B(x,t)\cos(\Omega t)+C(x,t)\sin(2\Omega t)+D(x,t)\cos(2\Omega t)+\cdots, \quad (5)$$

where $\overline{F},A,B,C,D$ are slowly varying amplitudes in time depending on the transverse coordinate. After substituting this Fourier expansion into Eq. (1) we obtain a system of equations for the functions $\overline{F},A,B,C,D$:

$$\overline{F}_t=\mu\overline{F}^*+(\gamma+i\delta_0)\overline{F}+i\frac{\delta_1 A}{2}-\frac{1}{4}(2A^2\overline{F}^*+4|A|^2\overline{F}+2B^2\overline{F}^*+4|B|^2\overline{F}+4|\overline{F}|^2\overline{F}+\cdots)+\overline{F}_{xx},$$

$$A_t-\Omega B=\mu A^*+(\gamma+i\delta_0)A+i\delta_1\overline{F}-\frac{i\delta_1 D}{2}-\frac{1}{4}(8A|\overline{F}|^2+4A^*\overline{F}^2+\cdots)+A_{xx},$$

$$B_t+A\Omega=\mu B^*+(\gamma+i\delta_0)B+\frac{i\delta_1 C}{2}-\frac{1}{4}(8B|\overline{F}|^2+4B^*\overline{F}^2+\cdots)+B_{xx},$$

$$C_t-2\Omega D=\mu C^*+(\gamma+i\delta_0)C+\frac{i\delta_1 B}{2}-\frac{1}{4}(8C|\overline{F}|^2+4C^*\overline{F}^2+\cdots)+C_{xx},$$

$$D_t+2\Omega C=\mu D^*+(\gamma+i\delta_0)D-\frac{i\delta_1 A}{2}-\frac{1}{4}(2B^2\overline{F}^*+4|B|^2\overline{F}+\cdots)+D_{xx}. \quad (6)$$

Inspecting these equations one finds the following first-order dependencies:

$$B\approx-\frac{i\delta_1}{\Omega}\overline{F},\quad D\approx-\frac{i\delta_1 B}{4\Omega},\quad A\sim\frac{1}{\Omega^2},\quad C\sim\frac{1}{\Omega^3}, \quad (7)$$

which lead to an ansatz of the form

$$A=\frac{a_1}{\Omega^2}+\frac{a_2}{\Omega^4},\quad B=\frac{b_1}{\Omega}+\frac{b_2}{\Omega^3},\quad C=\frac{c_1}{\Omega^3}+\frac{c_2}{\Omega^5},$$

$$D=\frac{d_1}{\Omega^2}+\frac{d_2}{\Omega^4}.$$

Using this ansatz the unknown quantities a,b,c,d can be determined as

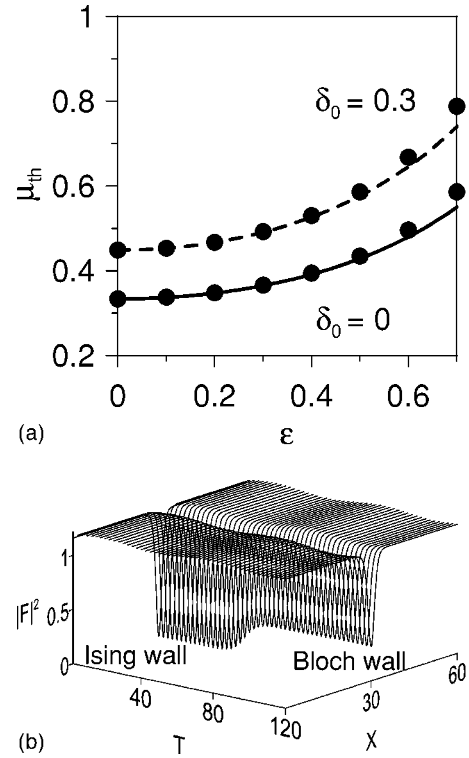

FIG. 1. Transition of an Ising to a Bloch wall for rapid periodically detuning $\delta=\delta_0+\delta_1\sin(\Omega t)$. (a) Threshold of the Ising-Bloch bifurcation as a function of the strength of the periodic perturbation $\epsilon=\delta_1/\Omega$, lines; averaged Eq. (9), dots; numerical simulation of Eq. (1); parameters, $\gamma=1$, $\delta_0=0,0.3$, $\Omega=10$. (b) Evolution of the order parameter F for $\mu=0.4$, $\gamma=1$, $\delta_0=0$, $\delta_1=7$, $\Omega=10$; initial condition, Ising wall.

$$a_1=i\delta_1\overline{F}_t+i\mu\delta_1\overline{F}^*-i\gamma\delta_1\overline{F}+\delta_0\delta_1\overline{F}+i\delta_1|\overline{F}|^2\overline{F}-i\delta_1\overline{F}_{xx},$$

$$b_1=-i\delta_1\overline{F},\quad c_1=-i\delta_1 A/4,\quad d_1=-\delta_1^2\overline{F}/4. \quad (8)$$

Finally, Eqs. (5)–(7) result in an amplitude equation for the redefined, averaged field $\tilde{F}+(1+\epsilon^2/2)^{1/2}\overline{F}$ of the form

$$\tilde{F}_t=\mu\frac{1-\epsilon^2/2}{1+\epsilon^2/2}\tilde{F}^*+(\gamma+i\delta_0)\tilde{F}-|\tilde{F}|^2\tilde{F}+\tilde{F}_{xx}$$

$$\approx\mu(1-\epsilon^2)\tilde{F}^*+(\gamma+i\delta_0)\tilde{F}-|\tilde{F}|^2\tilde{F}+\tilde{F}_{xx}, \quad (9)$$

where $\epsilon^2=\delta_1^2/\Omega^2$.

Equation (9) shows that a rapidly oscillating detuning effectively decreases the parametric forcing of the averaged field amplitude. For rather large oscillating perturbations the parametric forcing may even vanish, causing a recovery of the above-mentioned continuous phase symmetry. Moreover, for a purely oscillating detuning $\delta_0=0$, $\delta_1\neq 0$ the nongradient GL Eq. (1) transforms into an averaged gradient equation. Thus, in this case there is no averaged net velocity of Bloch walls. Figure 1(b) shows the evolution of an Ising wall for such a situation beyond the Ising-Bloch bifurcation point. The Ising wall destabilizes and transforms into a Bloch wall without net velocity. In Fig. 1(a) we compare the threshold of the Ising-Bloch bifurcation obtained by the averaged model

[Eq. (9)] with the results of numerical simulations of the basic Eq. (1). A very good agreement is obtained even for rather large perturbations ϵ.

Now, let us briefly discuss the influence of rapid oscillatory perturbations of the linear gain $\gamma=\gamma_0+\gamma_1\sin(\Omega t)$ on the Ising-Bloch transition. Similarly to the above procedure, an averaged GL equation

$$\overline{F_t} = \mu\overline{F^*} + (\gamma_0 + i\delta)\overline{F} - (1+2\epsilon^2)|\overline{F}|^2\overline{F} + \overline{F_{xx}} \tag{10}$$

can be obtained. Here, oscillatory perturbations lead only to a decrease of the amplitude of both types of walls leaving the bifurcation point of the Ising-Bloch transition unchanged.

It should be noted that the model of Eq. (1) is a nongradient system, the dynamics of which can be analyzed by means of other approaches, e.g., using the free energy or Lyapunov potential [20]. The comparison of such methods with the approach of the averaged GL equation requires separate considerations.

III. RANDOM MODULATIONS

In this section we study the influence of random parameter modulations on the Ising-Bloch transition. We restrict ourselves to perturbations of the detuning parameter $\delta=\delta(t)$, which exhibit the property of white noise $\langle\delta(t)\delta(t')\rangle = 2\sigma^2\delta_D(t-t')$, $\langle\delta(t)\rangle=0$. $\delta_D(t)$ is the Dirac delta function and $\langle\cdots\rangle$ denotes the averaging procedure over all realizations of the random process.

A. Dynamics of the first moment and transition threshold

Here, the equation for the first moment of the order parameter F is derived. By means of this averaged amplitude equation we study the influence of noise on the Ising-Bloch transition. First, we rewrite Eq. (1) in the form of two equations for the amplitude a and the phase ϕ of the complex order parameter $F=ae^{i\phi}$:

$$a_t = [\mu\cos(2\phi) + \gamma - \phi_x^2]a + a_{xx} - a^3, \tag{11}$$

$$\phi_t = \delta(t) - \mu\sin(2\phi) + \phi_{xx} + 2\phi_x a_x/a. \tag{12}$$

The Ising solution [Eq. (2)] is purely real, i.e., the phase vanishes, $\phi=0$. Therefore one may expect that under the influence of the time dependent perturbation $\delta(t)$ the phase of the perturbed Ising wall solution depends only on the time variable too. In this case the equation for the phase ϕ has the simple form

$$\phi_t = \delta(t) - \mu\sin(2\phi).$$

Small perturbations in $\delta(t)$ (we neglect large fluctuations) cause small values of the phase. Therefore we can approximately reduce Eq. (12) to a linear one,

$$\phi_t = \delta(t) - 2\mu\phi, \tag{13}$$

which has the following solution:

$$\phi(t) = e^{-2\mu t}\int_0^t e^{2\mu t'}\delta(t')dt'. \tag{14}$$

Looking for the solution of Eq. (11) in the form $a=\overline{a}+a_1$, where $\langle a\rangle=\overline{a}$, $\langle a_1\rangle=0$, we arrive at the system of equations

$$\overline{a_t} = (\mu\langle\cos 2\phi\rangle + \gamma)\overline{a} + \mu\langle a_1\cos 2\phi\rangle + \overline{a_{xx}} - \overline{a}^3 - 3\overline{a}\langle a_1^2\rangle, \tag{15}$$

$$\begin{aligned} a_{1t} &= \mu(\cos 2\phi - \langle\cos 2\phi\rangle)\overline{a} + \mu(a_1\cos 2\phi - \langle a_1\cos 2\phi\rangle) \\ &\quad + \gamma a_1 + a_{1xx} - a_1^3 - 3\overline{a}^2 a_1 + 3\overline{a}(a_1^2 - \langle a_1^2\rangle). \end{aligned} \tag{16}$$

As can be inferred from Eq. (16) the variable a_1 is at least of second order. Keeping only terms up to second order, we finally get the closed equation for the averaged amplitude,

$$\overline{a_t} = [\mu(1-2\langle\phi^2\rangle) + \gamma]\overline{a} + \overline{a_{xx}} - \overline{a}^3. \tag{17}$$

Using Eq. (14) we find that

$$\langle\phi^2\rangle = \frac{\sigma^2}{2\mu}(1-e^{-4\mu t}) \to \frac{\sigma^2}{2\mu} \tag{18}$$

holds. Thus the equation for the averaged amplitude coincides with the unperturbed GL equation after replacing μ by $\mu(1-2\langle\phi^2\rangle)\approx\mu-\sigma^2$. In our case of $\langle\delta\rangle=0$ the averaged GL equation is of gradient type whereas the original, stochastic GL equation is nongradient. Similar to the previous section the parametric forcing decreases for increasing strength of the random perturbation of the detuning. It can even vanish, producing an averaged GL equation without parametric forcing. If we apply the considerations in this section to harmonic perturbation $\delta(t)=\delta_1\sin(\Omega t)$, we get $\phi=(\delta_1/\Omega)[\exp(-2\mu t)-\cos(\Omega t)]\to-(\delta_1/\Omega)\cos(\Omega t)$ and thus

$$\langle\phi^2\rangle = \left\langle\frac{\delta_1^2}{2\Omega^2}\right\rangle \tag{19}$$

holds. Inserting this formula into above averaged Eq. (17) exactly the same result [Eq. (9)] as in the previous section is obtained. Even the change of the amplitude coincides if the approximation $\langle F\rangle=\langle a\exp(i\phi)\rangle\approx\overline{a}(1-\langle\phi^2\rangle)$ is used. The reason for the decrease of the parametric forcing originates from the fact that the perturbed Ising wall exhibits a nontrivial phase dependence. Of course, this nonzero phase influences the phase sensitive term in the parametric GL equation as can be observed, e.g., in Eq. (11). Thus, the effective parametric forcing becomes time dependent and it decreases after averaging.

With regard to the threshold of the Ising-Bloch bifurcation the results of numerical simulations of the full stochastic GL equation (1) coincide very well with the those of the averaged model [Eq. (17)] [see Fig. 2(a)]. A typical evolution of the order parameter F with rather strong δ noise is shown in Figs. 2(b) and 2(c). An Ising wall destabilizes and transforms into a Bloch wall. Even though such a Bloch wall exhibits no averaged net velocity there is a kind of Brownian motion, which will be studied in the next subsection.

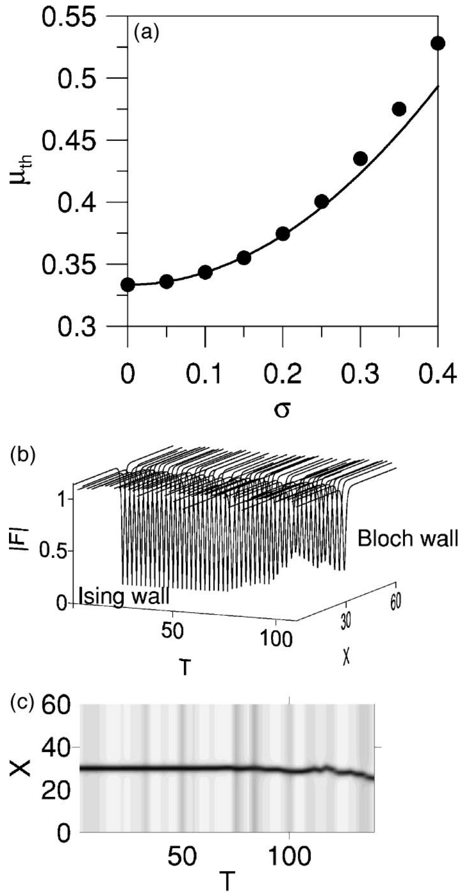

FIG. 2. Transition of an Ising to a Bloch wall for randomly varying detuning $\delta=\delta_1(t)$, $\langle\delta_1(t)\delta_1(t')\rangle=2\sigma^2\delta_D(t-t')$. (a) Threshold of the Ising-Bloch bifurcation as a function of the variance σ of the random perturbation, line; averaged Eq. (17), dots; numerical simulation by means of Eq. (1); parameter $\gamma=1$. (b) Surface plot of the evolution of the order parameter F for $\mu=0.35$, $\gamma=1$, $\sigma=0.4$; initial condition Ising wall. (c) Contour plot of the evolution of the order parameter F; same parameters as in (b).

Let us briefly discuss the role of strong fluctuations. As shown in the Appendix, the second moment for the phase is described by the expression

$$\langle\phi^2(t)\rangle \approx \frac{\sigma^2}{2\mu}(1-e^{-4\mu t}+4\mu t P_c), \tag{20}$$

which coincides with Eq. (18) for $4\mu t \ll P_c^{-1}$. Note, for an initial time interval $4\mu t \ll 1$ and not too small values of P_c we have the diffusion law for the phase $\langle\phi^2\rangle=Dt$ with the new diffusion coefficient $2\sigma^2(1+P_c)$ instead of $2\sigma^2$ for neglecting P_c. This will explain the numerical observations of increasing diffusion coefficients in the case of large fluctuations (see the numerical results for the diffusion of the domain wall in the next section).

Furthermore, the results of our investigations can be easily extended to a more involved noise model $\epsilon(x,t)$ with the properties $\langle\epsilon(x,t)\rangle=0$, $\langle\epsilon(x,t)\epsilon(y,t')\rangle=D(x-y,l_\epsilon)\delta(t-t')$, where l_ϵ is the correlation length. In this case our results are approximately valid after replacing σ^2 by $D(0)$. An analogous noise model is considered by Becker and Kramer [2] for the GL equation, where the influence of multiplicative noise on the bifurcation process has been studied. Within such a model one still has Markovian processes for the noise. Therefore, the theory of Markov proceses can be applied. The case of white noise in space and time is beyond the applicability of our theory and has to be considered separately.

B. Brownian motion of the Bloch wall

For temporal fluctuations of δ the Bloch wall will undergo a random motion. In order to analyze the character of this motion we derive the equation for the velocity of the Bloch wall.

For $\delta\sim\epsilon\ll 1$ we can apply the singular perturbation theory. For the order parameter F the following ansatz is used:

$$F = F_0(z) + \epsilon u_1(z,t) + \epsilon^2 u_2(z,t) + \cdots, \tag{21}$$

where $z=x-x_0(t)$ holds and F_0 is the Bloch solution in a comoving frame with velocity $v(t)$. The actual position x_0 of the Bloch wall is given by $x_0=\int_0^t v(t_1)dt_1$. The velocity $v(t)$ is assumed to obey the relation

$$v(t) = \epsilon v_1 + \epsilon^2 v_2 + \cdots. \tag{22}$$

After substituting all these expansions into Eq. (1) a system of equations for F_0, u_1, u_2 is obtained. Excluding the secular terms in the first order we find the following expression for the first-order velocity v_1:

$$v_1 = \delta(t)\frac{\int_{-\infty}^{\infty} dz[Y_0(y)X_{0z}(y) - X_0(y)Y_{0z}(y)]}{\int_{-\infty}^{\infty} dz[X_{0z}^2(y) + Y_{0z}^2(y)]}, \tag{23}$$

where $y=\sqrt{2\mu}z$ and X_0, Y_0 denote the real and the imaginary part of the unperturbed Bloch wall. If we take into account the shape of the Bloch wall [Eq. (3)] a simple relation for the position of the center of such a randomly perturbed domain wall can be deduced

$$x_0 = -A\int_0^t \delta(t_1)dt_1, \quad A = \frac{3\pi\sqrt{\gamma-3\mu}\sqrt{\mu+\gamma}}{2\sqrt{2\mu}(3\gamma-\mu)}. \tag{24}$$

For a constant detuning δ, this equation coincides with that found in [7]. For a random detuning the mean square of the Bloch wall displacement $\langle x_0^2\rangle$ is given by

$$\langle x_0^2\rangle = Dt, \quad D = 2A^2\sigma^2. \tag{25}$$

The mean square of the displacement of the domain wall increases linearly in time. Thus a Brownian motion of the domain wall occurs. The diffusion coefficient is proportional to the so-called chirality χ^2, where $\chi=\sqrt{\gamma-3\mu}$. Therefore the diffusion of the domain walls is slower close to the threshold of the Ising-Bloch transition. For small values of μ the relation $D\sim 1/\mu$ holds, i.e., in this case the diffusion is proportional to the square of the domain wall width.

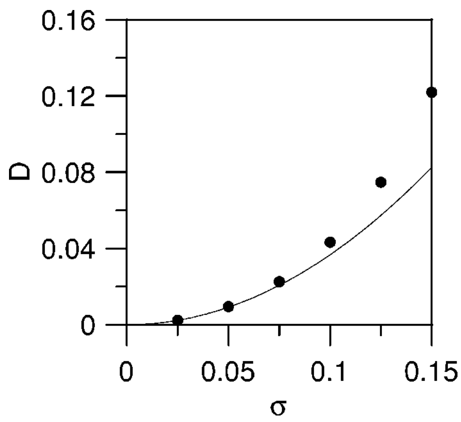

FIG. 3. Diffusion coefficient of the Brownian motion of a Bloch wall as a function of the variance of the randomly varying detuning $\delta=\delta_1(t)$, $\langle\delta_1(t)\delta_1(t')\rangle=2\sigma^2\delta_D(t-t')$. Solid line, theory; points, numerical simulation by means of Eq. (1); parameters $\gamma=1$, $\mu=0.25$.

Figure 3 shows the diffusion coefficient of Bloch walls as a function of the variance of the stochastic δ variations. The results of the numerical simulations of the full stochastic Ginzburg-Landau equation reasonably coincide with the above theory for low values of the noise. For larger noise strengths there is only a qualitative agreement (see Appendix). Probably this is due to the generation of new states by the strong noise. A similar threshold value for the noise intensity has been observed in the sine-Gordon equation, where the generation of small amplitude breathers has been observed [21–23].

IV. CONCLUSION

In conclusion, we have studied the Ising-Bloch transition in systems described by the parametric Ginzburg-Landau equation for periodically and randomly varying parameters in time. For rapidly modulated detuning and linear gain, averaged parametric GL equations have been derived. For temporal modulations of the linear detuning the threshold of the Ising-Bloch bifurcation is shifted in dependence on the strength of the perturbation. Temporal perturbations of detuning may even cause a change of the type of the nonlinear system, i.e., initially nongradient systems may effectively behave like gradient ones. On the contrary, rapid periodic perturbations of the linear gain do not change the critical point of the Ising-Bloch transition. This type of perturbation leads only to a decrease of the domain wall amplitudes.

For a white noise modulation of the detuning we studied the behavior of the first moment. In this case an averaged amplitude equation has been derived. Using this equation the shift of the bifurcation point was analyzed. Additionally, the connection between a periodically and randomly varying detuning parameter was examined. For randomly changing detunings Bloch walls perform a Brownian motion. The diffusion coefficient of this Brownian motion is derived for low amplitude noise. It is shown that the diffusion coefficient is proportional to the square of the chirality parameter. Thus, slow diffusion can be observed close to the critical point of the Ising-Bloch transition. All analytical predictions have been confirmed by numerical simulations of the parametric GL equation with periodic and random coefficients.

APPENDIX

In order to treat Eqs. (11) and (12) analytically we apply the two following assumptions, where the validity was checked numerically.

(i) For a random, time dependent detuning parameter the phase of the perturbed Ising wall depends on the time variable t only, or at least it is a slow function of the transverse variable x. Therefore, the terms ϕ_x and ϕ_{xx} in Eq. (12) can be omitted. The reduced equation exhibits the form

$$\phi_t = \delta(t) - \mu\sin(2\phi). \tag{A1}$$

(ii) For low intensity noise the phase is proportional to $\delta(t)$ and has a small magnitude. Thus, Eq. (A1) can be linearized. This assumption corresponds to the neglect of large noise fluctuations.

The influence of large fluctuations can be estimated in the following way. Let us define the time periods $\Delta t_n = t_{n+} - t_{n-}$ as such intervals (around random moments $t_n, t_{n-} \leqslant t_n \leqslant t_{n+}$, $n = 1,2,3,\ldots,N$) in which $\delta(t_n) > \delta_c$ occurs: δ_c is some critical value of the noise amplitude that is considered to be large (e.g., $\delta_c \geqslant \mu$). The value of the period Δt_n depends on a correlation time and it tends to zero for white noise. For such time intervals Eq. (A1) approximately reduces to $\phi_t(t_n) = \delta(t_n)$ leading to the following expression:

$$\langle\phi^2(t_{n+})\rangle \approx 2\sigma^2\Delta t_n. \tag{A2}$$

Thus, the value of the second moment of the phase can be written as [see Eq. (18)]

$$\langle\phi^2(t)\rangle \approx \frac{\sigma^2}{2\mu}\left(1 - e^{-4\mu t} + 4\mu\sum_{n=1}^{N}\Delta t_n\right), \quad t_N \leqslant t. \tag{A3}$$

The third term in the above parentheses describes the contribution of the large fluctuations and it can be rewritten as $\Delta\langle\phi^2(t)\rangle \approx 2\sigma^2 t P_L$, where $P_L = \Sigma_{n=1}^{N}\delta t_n/t$. The parameter P_L defines the relative time interval of large fluctuations, and for a large enough period of the time t it becomes a constant. In order to estimate P_L we assume a Gaussian distribution of δ according to

$$P(\delta) = \frac{1}{\sqrt{2\pi}\delta_0}\exp\left(-\frac{\delta^2}{2\delta_0^2}\right). \tag{A4}$$

In this case the probability for $\delta > \delta_c$ is given by

$$P_c = \int_{\delta_c}^{\infty} P(\delta)\delta = \frac{1}{\sqrt{\pi}}\int_{\delta_c/\sqrt{2}\delta_0}^{\infty} e^{-x^2}dx. \tag{A5}$$

For example, for $\delta_0 = 0.07$ and $\delta_c = 0.3$ we get $P_c \approx 10^{-5}$, and for $\delta_c = 0.4$ the value $P_c \approx 10^{-8}$ holds approximately. In the case of ergodic processes the relation $P_L = P_c$ results. Finally, the second moment of the phase reads as

$$\langle\phi^2(t)\rangle \approx \frac{\sigma^2}{2\mu}(1 - e^{-4\mu t} + 4\mu t P_c), \tag{A6}$$

which coincides with Eq. (18) for $4\mu t \ll P_c^{-1}$.

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